

Multi-access channels in quantum information theory.

A.E. Allahverdyan, D.B. Saakian

Yerevan Physics Institute

Abstract

The multi-access channels in quantum information theory are considered. Classical messages from independent sources, which are represented as some quantum states, are transported by a channel to one address. The messages can interact with each other and with external environment. After statement of problem and proving some general results we investigate physically important case when information is transported by states of electromagnetic field. One-way communication by noisy quantum channels is also considered.

1 Introduction

Physical ideas played important role as sources for information theory [1, 8]. Only in 15 years after discovery of information theory in works of Gordon, Lebedev and Levitin, and other researcher (for references see [1]) the general structure of information theory has been considered from physical point of view. Indeed quantum-mechanical and thermodynamical limitations are very important for correct consideration of information-theoretical models [1, 8].

More generally, quantum information theory contains two distinct types of problem. The first type describes transmission of classical information through a quantum channel (the channel can be noisy or noiseless). In such scheme bits encoded as some quantum states and only this states or its tensor products are transmitted. In the second case arbitrary superposition of this states or entanglement states are transmitted. In the first case the problems can be solved by methods of classical information theory, but in the second case

new physical representations are needed. In this work we investigate the problems of the first type.

In the almost all paper devoted to problems of physical information theory only one-way communication are considered- i.e., there is one input with some initial quantum ensemble and one output where quantum states are detected. Here quantum ensemble is some set of quantum states (which can be represented by corresponding density matrices) with corresponding probabilities. But in the practice multi-terminal communication schemes also are important. In this case there are several input for information and several output for detection. In this communication scheme messages are represented as some physical systems and there are interactions between the messages and external environment. For general discussion about multi-terminal classical information theory see [9].

Now about the concrete statement of problem. We can consider one output which receive information from two independent sources. The messages of these sources are represented as some quantum states. More exactly we can say that for any letter of some classical alphabet the concrete quantum state is generated. The quantum ensembles of two independent sources are

$$\rho^{(1)} = \sum_{\alpha} p_{\alpha}^{(1)} \rho_{\alpha}^{(1)}, \quad (1)$$

$$\rho^{(2)} = \sum_{\beta} p_{\beta}^{(2)} \rho_{\beta}^{(2)}. \quad (2)$$

After initial preparation the quantum states of (1, 2) penetrate through a quantum channel. In this channel there are interactions between the states of sources and an interaction with the environment. The concrete mechanisms of this interactions will be discussed later. The general effect of the noisy quantum channel can be described by a quantum evolution operator \hat{S} with kraussian representation

$$\hat{S}\rho = \sum_{\mu} A_{\mu}^{\dagger} \rho A_{\mu}, \quad \sum_{\mu} A_{\mu} A_{\mu}^{\dagger} = \hat{1}. \quad (3)$$

This operators must be linear, completely positive and trace-preserving [10, 6]. After interaction receiver has the states $\sigma_{\alpha\beta}^{(12)}$

$$\hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)}) = \sigma_{\alpha\beta}^{(12)} \quad (4)$$

These are states of quantum ensemble

$$\sigma = \sum_{\alpha, \beta} p_{\alpha} p_{\beta} \sigma_{\alpha\beta}^{(12)} \quad (5)$$

At the output of the channel receiver should separate and recognized the messages of sources. We can say that the receiver need some measurement procedure. It is important that the elements of (1,2) can be nonorthogonal or nonorthogonality can occur after action of (4). In this case for more optimal distinguishing between different quantum states we need generalized measurement procedure [4]. This type of measurement is represented by some nonorthogonal resolution of identity

$$\begin{aligned} \sum_{\gamma} E_{\gamma} &= 1 \\ E_{\gamma} &> 0, \quad E_{\gamma} = E_{\gamma}^{\dagger} \end{aligned} \quad (6)$$

If system with density matrix ρ is measured then probability of result γ is $\text{tr}(\rho E_{\gamma})$. In the [2] was shown that for distinguishing nonorthogonal states measurements like (6) more optimal than usual. In the output of the channel (4) as result of some generalized measurement like (6) we have the conditional probabilities

$$p(\gamma/\alpha\beta) = \text{tr}(E_{\gamma} \sigma_{\alpha\beta}^{(12)}) \quad (7)$$

This procedure is called decoding. With initial distributions

$$p(\alpha), \quad p(\beta) \quad (8)$$

(7) determined the definition of the binary multi-access channel in the classical sense [9, 8]. Let R_1, R_2 are the maximal quantities of information that the sources can transport in the regime of the reliable connection (it is the connection with small probability of error in the decoding). As was shown in the [9] such R_1, R_2 must satisfied the following conditions

$$R_1 < I(\alpha : \gamma/\beta), \quad R_2 < I(\beta : \gamma/\alpha), \quad (9)$$

$$R_1 + R_2 < I(\alpha \otimes \beta : \gamma). \quad (10)$$

Where

$$I(\alpha : \gamma/\beta) = \sum_{\alpha, \beta, \gamma} p(\alpha, \beta, \gamma) \ln \frac{p(\gamma/\alpha\beta)}{p(\gamma/\beta)}, \quad (11)$$

$$I(\alpha \otimes \beta : \gamma) = \sum_{\alpha, \beta, \gamma} p(\alpha, \beta, \gamma) \ln \frac{p(\gamma/\alpha\beta)}{p(\gamma)}. \quad (12)$$

The second value is usual mutual information between ensembles γ and $\alpha \otimes \beta$. The first value is called mutual-conditional information (mc-information). The mutual information of two ensembles is reduction of entropy of one ensemble if the second is observed. Mc-information has same meaning but after realization of the conditional ensemble. Obvious, that usual and famous formula of Shannon is particular case of the (9, 10).

The problem of physical information theory in this case is investigation the results of (9), (10) for physical important noisy channels.

The restriction (10) is indeed important because for independent initial distribution we have

$$I(\alpha : \gamma/\beta) + I(\beta : \gamma/\alpha) \geq I(\alpha \otimes \beta/\gamma) \quad (13)$$

In the general case the values (11), (12) can not be calculated explicitly. Therefore investigation the upper bounds for this values is very important problem. For the one-way communication such theorems was proved by A.Holevo [3]. The most general results in this direction was obtained in the [5]. We shall obtain the measurement independent upper bounds for (11), (12) by method of [5]. For this purpose we need some general results from quantum statistical physics.

Quantum relative entropy between two density matrices ρ_1, ρ_2 is defined as follows

$$S(\rho_1||\rho_2) = \text{tr}(\rho_1 \log \rho_1 - \rho_1 \log \rho_2). \quad (14)$$

This positive quantity was introduced by Umegaki [13] and characterizes the degree of 'closeness' of density matrices ρ_1, ρ_2 . The properties of quantum relative information were reviewed by M.Ohya [12]. Here only one basic property is mentioned.

$$S(\rho_1||\rho_2) \geq S(\hat{S}\rho_1||\hat{S}\rho_2). \quad (15)$$

The inequality was proved by Lindblad [11].

In the derivations of our theorems we use the method of the work [5]. We have for mc-information

$$I(\alpha : \gamma / \beta) = \sum_{\alpha, \beta, \gamma} p_{\alpha}^{(1)} p_{\beta}^{(2)} \text{tr}(E_{\gamma} \hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)})) \ln \frac{\text{tr}(E_{\gamma} \hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)}))}{\text{tr}(E_{\gamma} \hat{S}(\rho^{(1)} \otimes \rho^{(2)}))} \quad (16)$$

For any generalized measurement I_{γ} and density matrix ρ we define the following transformation

$$\rho \mapsto \{\text{tr}(I_{\gamma} \rho)\} \gamma. \quad (17)$$

In other words ρ is transformed to diagonal matrix with corresponding diagonal elements. This map is also general quantum evolution operator like (3). Now we shall use map (17) and theorem (15) for values like

$$\text{tr}[\hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)}) (\ln \hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)}) - \ln \hat{S}(\rho^{(1)} \otimes \rho^{(2)}))] \quad (18)$$

We get

$$I(\alpha : \gamma / \beta) \leq - \sum_{\alpha, \beta} p_{\alpha}^{(1)} p_{\beta}^{(2)} S(\hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)})) + \sum_{\beta} p_{\beta}^{(2)} S(\hat{S}(\rho^{(1)} \otimes \rho_{\beta}^{(2)})) \quad (19)$$

$$I(\alpha \otimes \beta : \gamma) \leq - \sum_{\alpha, \beta} p_{\alpha}^{(1)} p_{\beta}^{(2)} S(\hat{S}(\rho_{\alpha}^{(1)} \otimes \rho_{\beta}^{(2)})) + S(\hat{S}(\rho^{(1)} \otimes \rho^{(2)})) \quad (20)$$

These inequalities are very useful if general limits are developed.

Now about information transmission by electromagnetic field.

In practice the most important tool for information transmission is electromagnetic field. We briefly recall the connection between formalism which is described above and characteristics of an electromagnetic field in the vacuum or liner dielectric media [1]. Here information is connected with longitudinal characteristics of plane wave with fixed center-frequency and narrow bandwidth. But transverse (polarization, wave vector) characteristics are fixed. This statement of question is more or less realizable in the practice. Now about concrete statement of the problem. Two modes of the electromagnetic field with frequencies ω_1, ω_2 penetrate into the noisy channel. We have considered two kinds of communication channels, which are coherent channel and quadrature-squeezed (briefly

squeezed) channel. In the first case the inputs of the sources are coherent states of chosen field modes, and information is carried in the pattern of complex amplitude excitations. In the second case the inputs are squeezed states, and information is carried in the pattern of excitations of squeezed quadratures. Relative to a coherent states a quadrature-squeezed states have reduced quantum uncertainty in one quadrature component, called the squeezed quadrature. There is a corresponding increase in the uncertainty in the orthogonal quadrature component, called the amplified quadrature. For the case of coherent states information can be recovered by heterodyne detection, i.e., by measuring both quadratures of the mode. For the case of quadrature-squeezed states information is recovered by measuring of squeezed quadrature. In the channel the modes can interact together and with external thermostat. We assume that interaction between modes and interaction with the thermostat are linear (for discussion about realization of this type of interaction see [14]).

We describe our model by quantum Langevin equation [15] where rotating wave approximation is done.

$$i\dot{a}_1 = \omega_1 a_1 - i\frac{\gamma a_1}{2} + k a_2 + i\bar{F}_1, \quad (21)$$

$$i\dot{a}_2 = \omega_2 a_2 - i\frac{\gamma a_2}{2} + k a_1 + i\bar{F}_2 \quad (22)$$

Where a_1, a_2 are annihilation operators for the modes, k is the strength of cross mode interaction, γ is the damping constant (for simplicity we choose damping constant same for the modes), $\bar{F}_1(t), \bar{F}_2(t)$ are langevin forces (white noise). These equations are generated by Hamiltonian

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + k(a_1^\dagger a_2 + a_2^\dagger a_1) \quad (23)$$

Solution of these equations can be obtained immediately. For example

$$\begin{aligned} a_1 = & a_1(0)(\epsilon e^{i\phi_1 t} + (1 - \epsilon)e^{i\phi_2 t}) - a_2(0)\sqrt{\epsilon(1 - \epsilon)}(e^{i\phi_1 t} - e^{i\phi_2 t}) \\ & + \sqrt{\epsilon}e^{i\phi_1 t} \int_0^t e^{i\phi_1 t'} F_1(t') dt' + \sqrt{1 - \epsilon}e^{i\phi_2 t} \int_0^t e^{i\phi_2 t'} F_2(t') dt' \end{aligned} \quad (24)$$

Where:

$$\phi_{1,2} = -\lambda_{1,2} + i\frac{\gamma}{2}, \quad \lambda_{1,2} = \frac{\omega_1 + \omega_2 \pm \sqrt{(\omega_1 - \omega_2)^2 + 4k^2}}{2} \quad (25)$$

$$\frac{1 - \epsilon}{\epsilon} = \frac{(\lambda_1 - \omega_1)^2}{k^2} \quad (26)$$

$$\begin{aligned} \langle F_i^\dagger(t) F_i(t') \rangle &= \gamma \bar{n}_T(\lambda_{1,2}) \delta_{ij} \delta(t - t') \\ \bar{n}_T &= (\exp(\lambda_{1,2}/T) - 1)^{-1} \end{aligned} \quad (27)$$

$$\langle F_i(t) F_j^\dagger(t') \rangle = \gamma(\bar{n}_T(\lambda_{1,2}) + 1) \delta_{ij} \delta(t - t') \quad (28)$$

$$\langle F_i F_j \rangle = 0 \quad (29)$$

Now we assume that modes a_1, a_2 at the $t = 0$ (in the input of the channel) are in squeezed states with squeezing parameters r_1, r_2 (some of these parameters can be zero) and shifts $(\alpha_1, \alpha_2), (\beta_1, \beta_2)$ (for definition of squeezed states see [1]). In the output of the channel (at the moment t) one of the modes (for example a_1) is measured. The case when two modes are measured and decoders can interchange the available information can be obtained by simple modification of the final results. We shall consider only two type of measurement. In the case of heterodyning both component of the mode is measured simultaneously. In other words we have [1]

$$p(\gamma/\alpha\beta) = \frac{1}{\pi} \langle \gamma | \rho(t; a_1) | \gamma \rangle \quad (30)$$

Where $|\gamma\rangle$ is coherent state with shift γ , and $\rho(t; a_1)$ the density matrix of the mode a_1 at the moment t .

In the second case one component of the mode is measured. In this case

$$p(\gamma_1/\alpha\beta) = \langle \gamma_1 | \rho(t; a_1) | \gamma_1 \rangle \quad (31)$$

Where $|\gamma_1\rangle$ is eigenstate of the measuring component. Later we shall treat homodyne and heterodyne measurements simultaneously, and shall use symbol γ for the both cases.

Simple analysis shows [9] that capacities are maximized by gaussian input probabilities

$$\begin{aligned} p(\alpha) &\sim \exp\left(-\frac{1}{2}\alpha^T K_\alpha \alpha\right) \\ p(\beta) &\sim \exp\left(-\frac{1}{2}\beta^T K_\beta \beta\right) \end{aligned} \quad (32)$$

Where

$$\alpha^T = (\alpha_1, \alpha_2), \quad \beta^T = (\beta_1, \beta_2).$$

Here and in future gaussian integrals will be written up to multiplicative factor.

The Langevin equations (21,22) are linear and after time t gaussian state remains gaussian. It is convinient to calculate (31,30) in the formalism of Wigner function. The Wigner function of general gaussian state can be represented in the following form as function of means and square-means of this state

$$W(\text{Re}\gamma, \text{Im}\gamma) \sim \exp\left(-\frac{(\text{Re}\gamma - \langle \text{Re}a \rangle)^2}{2} K_{11} - \frac{(\text{Im}\gamma - \langle \text{Im}a \rangle)^2}{2} K_{22} - (\text{Re}\gamma - \langle \text{Re}a \rangle)(\text{Im}\gamma - \langle \text{Im}a \rangle) K_{12}\right) \quad (33)$$

Where

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}^{-1} = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} \quad (34)$$

$$\begin{aligned} c_{11} &= \langle (\text{Re}a)^2 \rangle - \langle \text{Re}a \rangle^2 \\ c_{22} &= \langle (\text{Im}a)^2 \rangle - \langle \text{Im}a \rangle^2 \\ c_{12} = c_{21} &= \frac{\langle \text{Re}a \text{Im}a + \text{Im}a \text{Re}a \rangle}{2} - \langle \text{Im}a \rangle \langle \text{Re}a \rangle \end{aligned} \quad (35)$$

As we see this channel is gaussian. Capacity region for two-terminal gaussian channel can be obtained exactly. If vectors α, β with distribution (32) are initial messages for the channel then we have for output vector γ

$$\gamma = A_1 \alpha + A_2 \beta + z. \quad (36)$$

Where A_1, A_2 are some matrices (we shall use the specific form of these matrices for homodyne and heterodyne measurements), and z is additive gaussian noisy vector with correlation matrix K^{-1}

$$\langle zz^T \rangle = K^{-1} \quad (37)$$

In this case we get

$$I(\gamma : \alpha \otimes \beta) = \ln \det(1 + K A_1 K_\alpha^{-1} A_1^T + K A_2 K_\beta^{-1} A_2^T)$$

$$\begin{aligned}
I(\gamma : \alpha/\beta) &= \ln \det(1 + KA_1K_\alpha^{-1}A_1^T) \\
I(\gamma : \beta/\alpha) &= \ln \det(1 + KA_2K_\beta^{-1}A_2^T)
\end{aligned} \tag{38}$$

After some not hard but tedious calculations we come to the concrete results. The first case is heterodyne measurement of the mode 1 at the moment t with

$$r_1 = 0, \quad r_2 = 0. \tag{39}$$

In this case we should choose the following initial distributions

$$K_\alpha^{-1} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \quad K_\beta^{-1} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \tag{40}$$

With

$$x = \frac{1}{2}\bar{n}_1, \quad y = \frac{1}{2}\bar{n}_2. \tag{41}$$

Where \bar{n}_1, \bar{n}_2 are mean photon number of the initial distributions. In this case we have from general formulas

$$\begin{aligned}
I(\gamma : \alpha \otimes \beta) &= \ln \left(1 + \frac{\bar{n}_1 e^{-\gamma t} + 2(\bar{n}_2 - \bar{n}_1)\epsilon(1 - \epsilon)e^{-\gamma t}(1 - \cos(t(\lambda_1 - \lambda_2)))}{\frac{1}{2}(e^{-\gamma t} + \Psi + 1)} \right) \\
I(\gamma : \alpha/\beta) &= \ln \left(1 + \frac{\bar{n}_1 e^{-\gamma t}(1 - 2\epsilon(1 - \epsilon)(1 - \cos(t(\lambda_1 - \lambda_2))))}{\frac{1}{2}(e^{-\gamma t} + \Psi + 1)} \right)
\end{aligned} \tag{42}$$

And for Ψ we have

$$\Psi = (1 - e^{-\gamma t})(2\epsilon\bar{n}_T(\lambda_1) + 2(1 - \epsilon)\bar{n}_T(\lambda_2) + 1) \tag{43}$$

We don't write expression for $I(\beta : \gamma/\alpha)$ if written formula are sufficient for understanding the behavior of this quantity.

For $t = 0$ these information measures coincide with well known expression for capacity of coherent state channel. For large t (42) tend to zero. But as we see the decay is not monotonic: the additional oscillation occur due to interaction between different modes. If we take $\epsilon = 1$ we come to one-terminal channel with gaussian noise. In this case capacity capacity monotonically tends to zero.

Now about homodyning with arbitrary r . In this case we should choose as initial distributions

$$K_{\alpha}^{-1} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \quad K_{\beta}^{-1} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \quad (44)$$

It is product of gaussian distribution for measuring component and delta-function for other component. With the choice of (44) we have optimal distribution of input energy. In this case we have the following connection between dispersion of the distribution and mean photon number [1]

$$x = \bar{n}_1 - \text{sh}^2 r_1, \quad y = \bar{n}_2 - \text{sh}^2 r_2. \quad (45)$$

In the first we consider one-terminal communication with arbitrary squeezing parameter r . In this case we have

$$I(\alpha : \gamma_1) = \frac{1}{2} \ln \left(1 + \frac{\cos^2(\lambda t)(\bar{n} - \text{sh}^2 r)}{\frac{1}{4}e^{-2r} + 2 \sin^2(\lambda t) \text{sh}(2r) + C} \right) \quad (46)$$

For given \bar{n} optimal squeezing parameter is determined by the following formula

$$e^{2r} = \frac{\sqrt{\cos^4(\lambda t) + C^2 + 4C(2\bar{n} + 1) \cos^2(\lambda t)} - \cos^2(\lambda t)}{C} \quad (47)$$

Where

$$C = (e^{\gamma t} - 1)(2\bar{n}_T + 1) \quad (48)$$

As we see optimal time-dependent squeezing parameter tends to zero for large t . It is well known that for zero t squeezed states are more effective than coherent one [1]. Indeed we have

$$I = \ln(1 + \bar{n}) \quad (49)$$

for coherent states and

$$I = \ln(1 + 2\bar{n}) \quad (50)$$

for squeezed states. As we see noise beat usefulness of squeezed states, and for optimal squeezing parameter we have (47).

Now about general case for homodyne measurement. We get

$$I(\alpha : \gamma_1/\beta) = \frac{1}{2} \ln \left(1 + \frac{(\bar{n}_1 - \text{sh}^2 r_1)u_1^2}{\frac{1}{4}(u_1^2 e^{-2r_1} + u_2^2 e^{2r_1}) + \frac{1}{4}(v_1^2 e^{-2r_2} + v_2^2 e^{2r_2}) + \frac{1}{4}\Psi} \right) \quad (51)$$

$$I(\alpha \otimes \beta : \gamma_1) = \frac{1}{2} \ln \left(1 + \frac{(\bar{n}_1 - \text{sh}^2 r_1)u_1^2 + (\bar{n}_2 - \text{sh}^2 r_2)v_1^2 + \frac{1}{4}(u_1^2 e^{-2r_1} + u_2^2 e^{2r_1}) + \frac{1}{4}(v_1^2 e^{-2r_2} + v_2^2 e^{2r_2}) + \frac{1}{4}\Psi}{\frac{1}{4}(u_1^2 e^{-2r_1} + u_2^2 e^{2r_1}) + \frac{1}{4}(v_1^2 e^{-2r_2} + v_2^2 e^{2r_2}) + \frac{1}{4}\Psi} \right) \quad (52)$$

Where

$$\begin{aligned} u_1 &= e^{-\gamma t/2} (\epsilon \cos(\lambda_1 t) + (1 - \epsilon) \cos(\lambda_2 t)) \\ u_2 &= -e^{-\gamma t/2} (\epsilon \sin(\lambda_1 t) + (1 - \epsilon) \sin(\lambda_2 t)) \\ v_1 &= -e^{-\gamma t/2} \sqrt{\epsilon(1 - \epsilon)} (\cos(\lambda_1 t) - \cos(\lambda_2 t)) \\ v_2 &= e^{-\gamma t/2} \sqrt{\epsilon(1 - \epsilon)} (\sin(\lambda_1 t) - \sin(\lambda_2 t)) \end{aligned} \quad (53)$$

We can maximize by r_1, r_2 the information measure $I(\alpha : \gamma/\beta)$ (as we know it is information transmitted by user 1), and after this information measure $I(\alpha \otimes \beta : \gamma) - I(\alpha : \gamma/\beta)$ for user 2. The analysis show that situation with one-terminal channel is conserved: there are optimal r_1, r_2 which tend to zero for large t . The second mode introduce additional source for noise.

We have considered the noisy binary-access quantum-mechanical channel, and compute capacities of this channel. The general theorems are proved which connect capacities of the channel with some statistic-mechanical functions. It is shown that second user acts as noise and can significantly reduce the capacity of the first user. After this the information transfer by coherent and squeezed states of an electromagnetic field is discussed. Squeezed states loss their optimality under action of noise and optimal squeezing parameter tends to zero when time tends to infinity.

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